ECS 203 - Part 2B - For CPE2

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CHAPTER 9

# AC Power Analysis

Our effort in ac circuit analysis so far has been focused mainly on calculating voltage and current. The major concern in this chapter is power analysis.

9.0.4. Power is the most important quantity in electric utilities, electronic and communication systems because such systems involve transmission of power (or energy) from one point to another.

Every industrial and household electrical device (every fan, motor, lamp, pressing iron, TV, personal computer) has a **power rating** that indicates how much power the equipment requires; exceeding the power rating can do permanent damage to an appliance.

9.0.5. The most common form of electric power is 50-Hz (Thailand) or 60-Hz (United States) ac power. The choice of ac over dc allowed high-voltage power transmission from the power generating plant to the consumer. (DC attenuation is high.)

## 9.1. Instantaneous Power

Definition 9.1.1. The **instantaneous power** $p(t)$ absorbed by an element is the product of the instantaneous voltage $v(t)$ across the element and the instantaneous current $i(t)$ through it.

Assuming the **passive sign convention** as shown in Figure 1,

$$\boxed{p(t) = v(t)i(t).}$$

The instantaneous power is the power at any instant of time. It is **the rate at which an element *absorbs* energy**.

9.1.2. Consider the general case of instantaneous power absorbed by an arbitrary combination of circuit elements under sinusoidal excitation.

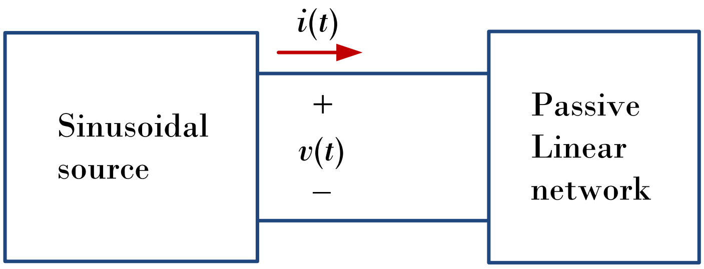



FIGURE 1. Sinusoidal source and passive linear circuit

Let the voltage and current at the terminals of the circuit be

$$v(t) = V_m \cos(\omega t + \theta_v)$$

and

$$i(t) = I_m \cos(\omega t + \theta_i)$$

where $V_m$ and $I_m$ are the amplitudes, and $\theta_v$ and $\theta_i$ are the phase of the voltage and current, respectively. The instantaneous power absorbed by the circuit is

$$p(t) = v(t)i(t) = V_m I_m \cos(\omega t + \theta_v)\cos(\omega t + \theta_i) \tag{9.4}$$

$$= V_m I_m \frac{e^{j(\omega t+\theta_v)} + e^{-j(\omega t+\theta_v)}}{2} \frac{e^{j(\omega t+\theta_i)} + e^{-j(\omega t+\theta_i)}}{2} \tag{9.5}$$

$$= V_m I_m \frac{1}{4}\left(e^{j(2\omega t+\theta_v+\theta_i)} + e^{j(\theta_i-\theta_v)} + e^{j(\theta_v-\theta_i)} + e^{-j(2\omega t+\theta_v+\theta_i)}\right) \tag{9.6}$$

$$= V_m I_m \frac{1}{2}\left(\frac{e^{j(\theta_v-\theta_i)} + e^{j(\theta_i-\theta_v)}}{2} + \frac{e^{j(2\omega t+\theta_v+\theta_i)} + e^{-j(2\omega t+\theta_v+\theta_i)}}{2}\right) \tag{9.7}$$

$$= V_m I_m \frac{1}{2}\left(\cos\left(\theta_v - \theta_i\right) + \cos\left(2\omega t + \theta_v + \theta_i\right)\right). \tag{9.8}$$

Alternatively, from (9.4), we can apply the trigonometric identity

$$\cos A \cos B = \frac{1}{2}\{\cos(A - B) + \cos(A + B)\}$$

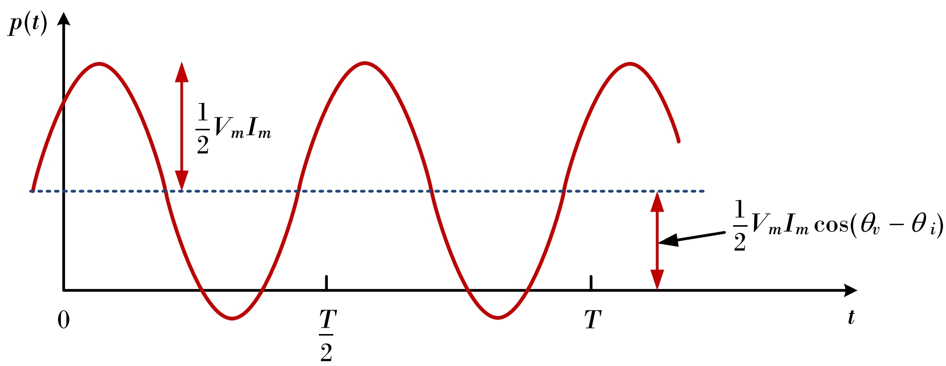
to directly arrive at (9.8) which is

$\theta_i - \theta_v$

$$(9.9)\qquad p(t) = \underbrace{\frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)}_{\text{constant term}} + \underbrace{\frac{1}{2}V_m I_m \cos(2\omega t + \theta_v + \theta_i)}_{\text{time-dependent term}}.$$

9.1.3. The instantaneous power, when expressed in the form of (9.9), has two parts:

(a) **First Part**: a constant or time-independent term. Its value depends on the phase difference between the voltage and the current.
(b) **Second Part**: a sinusoidal function whose angular frequency is $2\omega$, which is twice the angular frequency of the voltage or current.




9.1.4. Consider the sketch of $p(t)$, we observe that

(a) $p(t)$ is periodic and has a period of $T_o = \frac{T}{2}$, where $T = \frac{2\pi}{\omega}$ is the period of the voltage and the current
(b) $p(t)$ may become positive for some part(s) of each cycle and negative for the rest of the cycle.
   - When $p(t)$ is positive, power is absorbed by the circuit.
   - When $p(t)$ is negative, power is absorbed by the source.
     - In this case, power is transferred from the circuit to the source.
     - This is possible because of the storage elements (capacitors and inductors) in the circuit.

real power

## 9.2. Average Power

P

This formula works even for non-periodic signals

The instantaneous power changes with time and is therefore difficult to measure. The average power is more convenient to measure.

DEFINITION 9.2.1. The **average power** (of any periodic signal) is the average of the instantaneous power over one period.

In general, $P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} p(t)\,dt$

for periodic signal with period T

Thus, the average power is given by

$$P = \frac{1}{T}\int_0^T p(t)\,dt = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i).$$

for sinusoidal sigals

- Since $\cos(\theta_v - \theta_i) = \cos(\theta_i - \theta_v)$, what is important is the difference in the phases of the voltage and the current.
- Note also that $p(t)$ is time varying while $P$ does not depend on time.

9.2.2. Using the phasor forms of $v(t)$ and $i(t)$, which are $\mathbf{V} = V_m \angle \theta_v$ and $\mathbf{I} = I_m \angle \theta_i$, we obtain

complex conjugation

$$\boxed{P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) = \frac{1}{2} \mathbf{Re}\{\mathbf{V}\mathbf{I}^*\}.}$$

Recal DC

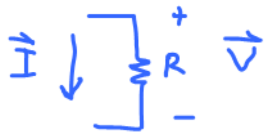
$P = VI$

9.2.3. Two special cases:

**Case 1:** When $\theta_v = \theta_i$, the voltage and the current are in phase. This implies a **purely resistive circuit** or resistive load $R$, and

$$P = \frac{1}{2} V_m I_m$$

$\vec{V} = \vec{I} R$




This shows that **a purely resistive circuit (e.g. resistive load (R)) absorbs power all times**.

**Case 2:** When $\theta_v - \theta_i = \pm 90°$, we have a **purely reactive circuit**, and

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$$P = \frac{1}{2} V_m I_m \cos(90°) = 0$$

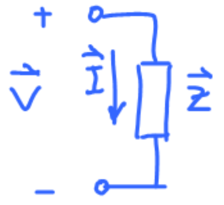
showing that **a purely reactive circuit (e.g. a reactive load $L$ or $C$) absorbs no average power**.

Recall the DC-version of the power formula:

$P = VI = \frac{V^2}{R} = I^2 R$

$|\vec{Z}| = \sqrt{30^2 + (-70)^2}$

EXAMPLE 9.2.4. Calculate the average power absorbed by an impedance $\mathbf{Z} = 30 - j70\ \Omega$ when a voltage $\mathbf{V} = 120\angle 0°$ is applied across it.



$P = \frac{1}{2}\mathrm{Re}\{\vec{V}\vec{I}^*\} = \frac{1}{2}\mathrm{Re}\{\vec{V}\left(\frac{\vec{V}}{\vec{Z}}\right)^*\} = \frac{1}{2}\mathrm{Re}\{\vec{V}\frac{\vec{V}^*}{\vec{Z}^*}\} = \frac{1}{2}\mathrm{Re}\{\frac{|\vec{V}|^2}{\vec{Z}^*}\}$

$= \frac{|\vec{V}|^2}{2}\mathrm{Re}\{\frac{1}{\vec{Z}^*}\} = \frac{|\vec{V}|^2}{2}\mathrm{Re}\{\frac{\vec{Z}}{|\vec{Z}|^2}\} = \frac{1}{2}\frac{|\vec{V}|^2}{|\vec{Z}|^2}\mathrm{Re}\{\vec{Z}\} = \frac{1}{2}\frac{120^2}{30^2+70^2} \times 30$

$= 37.24\ W$

EXAMPLE 9.2.5. A current $\mathbf{I} = 10\angle 30°$ flows through an impedance $\mathbf{Z} = 20\angle -22°\ \Omega$. Find the average power delivered to the impedance.

$P = \frac{1}{2}\mathrm{Re}\{\vec{V}\vec{I}^*\} = \frac{1}{2}|\vec{I}|^2\mathrm{Re}\{\vec{Z}\} = \frac{1}{2} \times 10^2 \times \underline{20 \times \cos(-22°)}$

$= 927\ W$

This part can be found directly from your calculator. (convert $20\angle 22°$ to rectangular form)

9.2.6. From Ohm's law, we have two more formula.

(a) The average power absorbed by an impedance $\mathbf{Z}$ when a voltage $\mathbf{V}$ is applied across it is

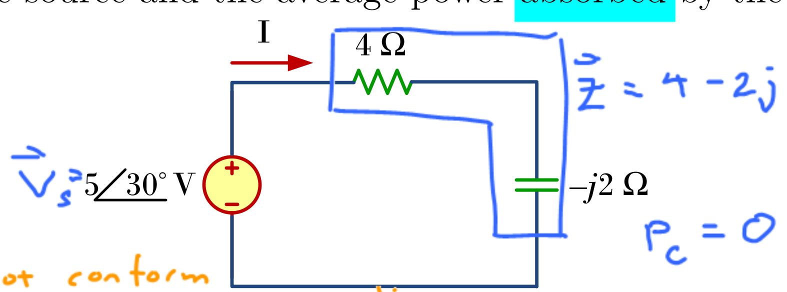
$$P = \frac{1}{2}\mathrm{Re}\{\mathbf{V}\mathbf{I}^*\} = \frac{1}{2}\mathrm{Re}\{\mathbf{V}\frac{\mathbf{V}^*}{\mathbf{Z}^*}\} = \frac{1}{2}|\mathbf{V}|^2\mathrm{Re}\{\frac{1}{\mathbf{Z}^*}\} = \frac{1}{2}|\mathbf{V}|^2\frac{\mathrm{Re}\{\mathbf{Z}\}}{|\mathbf{Z}|^2}.$$

(b) The average power absorbed by an impedance $\mathbf{Z}$ when a current $\mathbf{I}$ flows through it is

$$P = \frac{1}{2}\mathrm{Re}\{\mathbf{V}\mathbf{I}^*\} = \frac{1}{2}\mathrm{Re}\{\mathbf{I}\mathbf{Z}\mathbf{I}^*\} = \frac{1}{2}|\mathbf{I}|^2\mathrm{Re}\{\mathbf{Z}\}.$$

EXAMPLE 9.2.7. For the circuit shown below, find the average power supplied by the source and the average power absorbed by the resistor.

(a) (b)




$\vec{Z} = 4 - 2j$

$P_C = 0$

(a)

$\vec{I} = \frac{\vec{V}_s}{\vec{Z}}$

Direction of $\vec{I}$ and polarity of $\vec{V}_s$ do not conform with the passive sign convention

$P_{V_s} = -\frac{1}{2}\mathrm{Re}\{\vec{V}_s\vec{I}^*\} = -\frac{1}{2}\mathrm{Re}\{\vec{V}_s\left(\frac{\vec{V}_s}{\vec{Z}}\right)^*\} = -\frac{1}{2}\frac{|\vec{V}_s|^2}{|\vec{Z}|^2}\mathrm{Re}\{\vec{Z}\}$

$= -\frac{1}{2} \times \frac{5^2}{4^2+2^2} \times 4 = -2.5\ W \Rightarrow$ absorb $-2.5\ W$

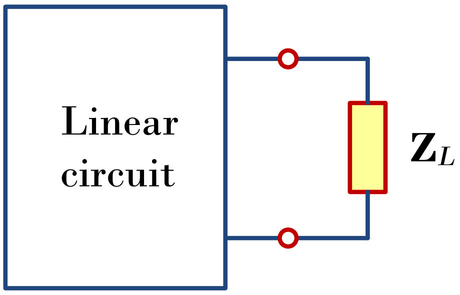
$\equiv$ supply $+2.5\ W$

(b) We know that $P_C = 0$. So, all average power supplied by the source must be absorbed by the resistor. So, $P_R = 2.5\ W$

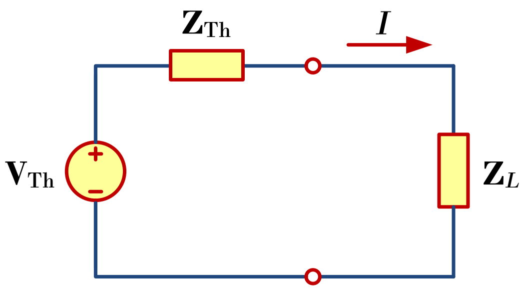
## 9.3. Maximum Average Power Transfer

9.3.1. Recall that, in an earlier chapter, we solved the problem of maximizing the power delivered by a power-supplying resistive network to a load $R_L$. Representing the circuit by its Thevenin equivalent, we proved that the maximum power would be delivered to the load if the load resistance is equal to the Thevenin resistance $R_L = R_{Th}$.

9.3.2. **Optimal Load Impedance**: We now extend that result to ac circuits.




Consider an ac circuit which is connected to a load $\mathbf{Z_L}$ and is represented by its Thevenin equivalent.




In a rectangular form, the Thevenin impedance $\mathbf{Z_{Th}}$ and load impedance $\mathbf{Z_L}$ are

$$\mathbf{Z_{Th}} = R_{Th} + jX_{Th}$$
$$\mathbf{Z_L} = R_L + jX_L$$

The current through the load is

$$\mathbf{I} = \frac{\mathbf{V_{Th}}}{\mathbf{Z_L} + \mathbf{Z_{Th}}},$$

and the average power delivered to the load is

$$P = \frac{1}{2}|\mathbf{I}|^2 \mathrm{Re}\{\mathbf{Z_L}\} = \frac{1}{2}|\mathbf{I}|^2 R_L = \frac{1}{2}\left|\mathbf{V_{Th}}^2\right| \frac{R_L}{(R_L + R_{Th})^2 + (X_L + X_{Th})^2}$$

Our objective is to adjust the load parameter $R_L$ and $X_L$ so that $P$ is maximum. To do this we set $\frac{\partial P}{\partial R_L}$ and $\frac{\partial P}{\partial X_L}$ equal to zero.

Setting $\frac{\partial P}{\partial X_L} = 0$ gives

$$X_L = -X_{Th}.$$

Setting $\frac{\partial P}{\partial R_L} = 0$ gives

$$R_L = \sqrt{R_{Th}^2 + (X_{Th} + X_L)^2}$$

Hence, to get the maximum **average power** transfer, the load impedance $\mathbf{Z_L}$ must selected so that

$$X_L = -X_{Th} \text{ and } R_L = R_{Th},$$

i.e.,

$$\boxed{\mathbf{Z_L} = R_L + jX_L = R_{Th} - jX_{Th} = \mathbf{Z^*_{Th}}}$$

That is, **for the maximum average power transfer, the load impedance $\mathbf{Z_L}$ must be equal to the complex conjugate of the Thevenin impedance $\mathbf{Z_{Th}}$.**

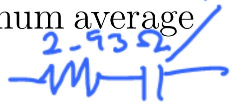
When $\mathbf{Z_L}$ is set to be $\mathbf{Z^*_{Th}}$, the corresponding maximum average power that can be transferred to the load is

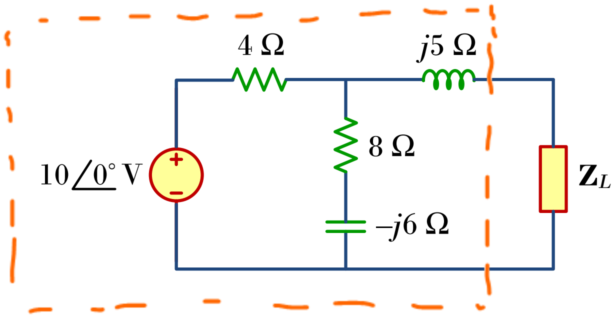
$$\boxed{P_{\max} = \frac{|\mathbf{V_{Th}}|^2}{8R_{Th}}.}$$

$\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$

$\frac{1}{j\omega C} = \frac{-j}{\omega C}$

Example 9.3.3. Determine the load impedance $\mathbf{Z}_L$ that maximizes the average power drawn from the circuit below. What is the maximum average power?

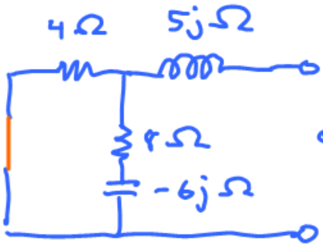



$\vec{Z}_L = 2.93 - 4.\overset{4}{\cancel{6}}7j\ \Omega$

$P_{max} = \frac{|\vec{V}_{Th}|^2}{8 R_{Th}} = 2.367\ W$

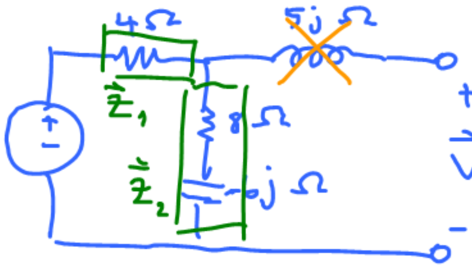
$\vec{Z}_{Th}$




$\leftarrow \vec{Z}_{Th} = \left((8-6j)//4\right) + 5j$

$= 2.93 + 4.\overset{4}{\cancel{6}}7j\ \Omega$

$\vec{V}_{Th}$




voltage divider

$\vec{V}_{Th} = \frac{\vec{Z}_2}{\vec{Z}_1 + \vec{Z}_2} \times 10 = \frac{8-6j}{4+8-6j} \times 10 = 7.33 - 1.33j$

9.3.4. **Optimal Purely Resistive Load**: In a situation in which the load must be purely real; that is $X_L$ must be 0. Then,

$$P = \frac{1}{2}|\mathbf{I}|^2 R_L = \frac{1}{2}\left|\mathbf{V}_{Th}^2\right| \frac{R_L}{(R_L + R_{Th})^2 + (X_{Th})^2}$$

Setting $\frac{\partial P}{\partial R_L} = 0$ gives

$$R_L = \sqrt{R_{Th}^2 + X_{Th}^2} = |\mathbf{Z_{Th}}|.$$

Hence, **for maximum average power transfer to a purely resistive load, the load impedance is equal to the magnitude of the Thevenin impedance $\mathbf{Z_{Th}}$**. In which case, the maximum average power is

$$P = \frac{1}{4}\left|\mathbf{V_{Th}}^2\right| \frac{1}{|\mathbf{Z_{Th}}| + R_{Th}}$$

Note that

$$|\mathbf{Z_{Th}}| + R_{Th} \geq R_{Th} + R_{Th} = 2R_{Th}$$

Hence,

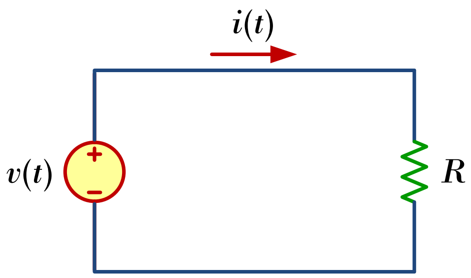
$$\frac{1}{4}\left|\mathbf{V_{Th}}^2\right| \frac{1}{|\mathbf{Z_{Th}}| + R_{Th}} \leq \frac{1}{8}\frac{\left|\mathbf{V_{Th}}^2\right|}{R_{Th}}.$$

## 9.4. Effective or RMS Value

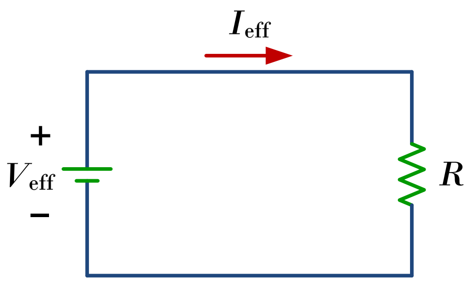
The idea of **effective value** arises from the need to measure the effectiveness of an ac voltage or current source in delivering power to a resistive load. We start by considering the more general case of periodic signals and then consider the special case of sinusoidal signals in 9.4.6.

DEFINITION 9.4.1. The **effective value $I_{\text{eff}}$ of a periodic current** $i(t)$ is the dc current that delivers the same average power to a resistor as the periodic current.

9.4.2. Consider the following "ac" and dc circuits,


our objective is to find the current $I_{\text{eff}}$ that will transfer the same power to the resistor $R$ as the sinusoid current $i$




The average power absorbed by the resistor in the "ac" circuit is

$$P = \frac{1}{T}\int_0^T i^2(t)Rdt = \frac{R}{T}\int_0^T i^2(t)dt.$$

Q: Where don't we have the factor of $\frac{1}{2}$?

A: It is hidden inside $\frac{1}{T}\int_0^T i^2(t)dt$ when $i(t)$ is a sinusoid.

The power absorbed by the resistor in the dc circuit is

$$P = I_{\text{eff}}^2 R.$$

$P = P$

root square

Equating the two expressions and solving for $I_{\text{eff}}$, we obtain

$$\boxed{I_{\text{eff}} = \sqrt{\frac{1}{T}\int_0^T i^2(t)\,dt}}$$

average (mean)

9.4.3. Similarly, the effective value of the periodic voltage is found in the same way as current; that is,

$$\boxed{V_{\text{eff}} = \sqrt{\frac{1}{T}\int_0^T v^2(t)\,dt}}$$

9.4.4. This indicates that the effective value is the square root of the mean (or average) of the square of the periodic signal. Thus, the effective value is often know as the **root mean square, or rms value** for short. We write

$$I_{\text{eff}} = I_{\text{rms}}, \quad V_{\text{eff}} = V_{\text{rms}}$$

9.4.5. Note that the rms value of a constant is the constant itself.

9.4.6. ***AC Circuit***: For a sinusoid $x(t) = X_m\cos(\omega t + \theta_x)$, the effective value or rms value is

$$X_{rms} = \sqrt{\frac{1}{T}\int_0^T X_m^2\cos^2(\omega t + \theta_x)\,dt} = \frac{X_m}{\sqrt{2}}.$$

In particular, for $i(t) = I_m\cos(\omega t + \theta_i)$ and $v(t) = V_m\cos(\omega t + \theta_v)$, we have

$$\boxed{I_{rms} = \frac{I_m}{\sqrt{2}} \text{ and } V_{rms} = \frac{V_m}{\sqrt{2}}.}$$

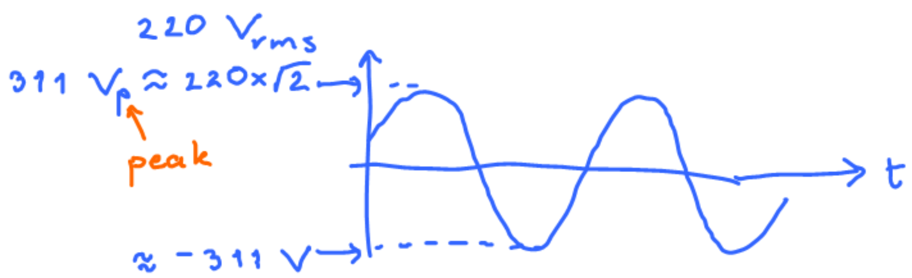
and the ***average power*** can be written in terms of the rms values as

$$\boxed{P = \frac{1}{2}V_m I_m\cos(\theta_v - \theta_i) = V_{rms}I_{rms}\cos(\theta_v - \theta_i)}.$$

9.4.7. Similarly, the average power absorbed by a resistor $R$ can be written as

$$P = I_{rms}V_{rms} = I_{rms}^2 R = \frac{V_{rms}^2}{R}.$$

Ex. Household voltage



## 9.5. Apparent Power and Power Factor

DEFINITION 9.5.1. The **apparent power** $S$ (in VA) is the product of the rms values of voltage and current.

$$S = \overset{(220V_{rms})}{V_{rms}} I_{rms}$$

$$P = \underbrace{\overbrace{V_{rms} I_{rms}}^{S} \cos(\theta_v - \theta_i)}_{pf}$$

Hence, the average power $P = S\cos(\theta_v - \theta_i)$.

DEFINITION 9.5.2. The **power factor** (pf) is the ratio of the average power to the apparent power.

$$\text{pf} = \frac{P}{S} = \cos(\theta_v - \theta_i).$$

Hence,

**average power** $P$ = **apparent power** $S$ × **power factor** pf.

The angle $\theta_v - \theta_i$ is called the **power factor angle** which is equal to the angle of the load impedance if $\mathbf{V} = V_m\angle\theta_v$ is the voltage across the load and $\mathbf{I} = I_m\angle\theta_i$ is the current through it. This is evident from the fact that

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{V_m\angle\theta_v}{I_m\angle\theta_i} = \frac{V_m}{I_m}\angle(\theta_v - \theta_i).$$

Alternatively, define $\mathbf{V_{rms}} = \frac{\mathbf{V}}{\sqrt{2}} = V_{rms}\angle\Theta_v$ and $\mathbf{I_{rms}} = \frac{\mathbf{I}}{\sqrt{2}} = I_{rms}\angle\Theta_i$. The impedance can then be written as

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{\mathbf{V_{rms}}}{\mathbf{I_{rms}}} = \frac{V_{rms}}{I_{rms}}\angle(\theta_v - \theta_i).$$

**The power factor is the cosine of the phase difference between the voltage and current. It is also the cosine of the angle of the load impedance.**

9.5.3. The value of the power factor pf ranges between 0 and 1.

- For a purely resistive load, the voltage and current are in phase, so that $\theta_v - \theta_i = 0$ and pf =1. This implies that the average power is equal to the apparent power .
- For a purely reactive load, $\theta_v - \theta_i = \pm 90°$. Hence, $pf = 0$. In this case the average power is zero.
- In between these two extreme cases, pf is said to be **leading** or **lagging**. **Leading power factor** means that current leads voltage which implies a capacitive load. **Lagging power factor** means that current lags voltage, implying an inductive load.

## 9.6. Complex Power

The **complex power** (in VA) $\mathbf{S}$ is the product of the rms voltage phasor and the complex conjugate of the rms current phasor. $\mathbf{S}$ is a complex quantity whose real part is the real or average power $P$ and imaginary part is the **reactive** power $Q$.

Complex power: $\mathbf{S}$

$$\mathbf{S} = P + jQ = \frac{1}{2}\mathbf{V}\mathbf{I}^* = \mathbf{V_{rms}}\mathbf{I^*_{rms}} = V_{rms}I_{rms}\angle(\theta_v - \theta_i) = I^2_{rms}\mathbf{Z} = \frac{V^2_{rms}}{\mathbf{Z}^*}.$$

Note that all previously studied quantities can be derived from the complex power. That is,

The apparent power $S$ is the magnitude of the complex power $\mathbf{S}$, i.e.,

$$S = |\mathbf{S}| = V_{rms}I_{rms} = \sqrt{P^2 + Q^2}.$$

The real or average power $P$ is

$$P = \mathrm{Re}\{\mathbf{S}\} = S\cos(\theta_v - \theta_i) = I^2_{rms}R.$$

The reactive power $Q$ is

$$Q = \mathrm{Im}\{\mathbf{S}\} = S\sin(\theta_v - \theta_i) = I^2_{rms}X.$$

The power factor pf is

$$\mathrm{pf} = \frac{P}{S} = \cos(\theta_v - \theta_i) = \cos(\text{phase of } \mathbf{S}).$$